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On the Transformation of Mass and Momentum Densities in Special Relativity[1]

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Abstract

By considering the mass and momentum densities of a point mass moving at uniform velocity, the known transformation of these densities from a representation in one inertial system to another is easily derived. The transformation is not linear in mass and momentum density, but the introduction of a dyadic stress density tensor gives a linear relation. The transformation is shown to hold for a general continuous mass distribution in which mass and momentum are conserved, provided a specific choice is made for the stress density tensor. This result contrasts with the particle viewpoint of matter in which only the divergence of the stress density tensor need be fixed so far as the transformation is concerned. A change of functions is made which greatly simplifies the transformations. The new functions are shown to represent a conserved fluid.

1. *Introduction*

In this paper, the transformation of representations of mass and momentum densities from one inertial system to another are investigated. Although the mass density transformation has been known for many years (Lorentz, 1927, sections 37-38; Møller, 1972, chap. 6), its application has been largely confined to the behavior of particles, where some subtleties are apt to be overlooked.

This investigation assumes a deterministic model of continuously distributed mass. For this purpose, either a macroscopic viewpoint is adopted, similar to that of electromagnetics, or this analysis is presumed to extend into the interior of molecules.

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The development here parallels modern work in electromagnetic theory (Elliott, 1966). The transformation itself is analogous to, but considerably more involved than, the charge density and current density transformation of electromagnetics (Stratton, 1941, pp. 74–80).

The paper and results are organized as follows: In section 2 the basic theory of special relativity that is necessary to the sequel is developed. This development also serves to introduce much of the notation that is used later.

In section 3 the special relativity transformations of the net charge, mass, and momentum of a body are used to derive transformations of the densities of these quantities, for bodies traveling at uniform velocities in an inertial frame.

The main results begin in section 4, in which the transformations of the preceding section are shown to be general ones, provided a certain specific choice of stress density tensor is made.

In section 5, the transformations are simplified by the introduction of new functions, and these functions, one a scalar, the other a vector, are shown to be related by a continuity equation.

2. *Preliminaries*

The results of the special theory of relativity that are of particular importance to the work to follow are now summarized and notation is introduced.

The transformation of velocity, because its development is so often linked to the motion of particles in introductory texts, is here presented in terms of parametric equations of motion, a viewpoint more in keeping with the continuum model of the following sections.

Special Relativity Coordinate Transformations. Consider two Cartesian coordinate systems, (x, y, z) and (x', y', z'), with the respective axes of the two systems aligned and the x' axis moving in the x direction with constant velocity v. For simplicity, let the times in the unprimed system, t, and in the primed system, t', both be taken to be zero when the origins of the two systems are coincident.

The Lorentz transformation, which relates the position and time of an event in one such inertial system to the position and time of the same event in the other inertial system, is

$$\begin{aligned} x' &= K(x - vt) \\ y' &= y \\ z' &= z \\ t' &= K[t - (vx/c^2)] \end{aligned} \tag{2.1}$$

where

$$K = 1/[\;-(v/c)^2]$$

The transformation from the primed to the unprimed system has the same form, with v replaced by $-v$.

Velocity Transformations. Let the parametric functions $X(t)$, $Y(t)$, $Z(t)$ express a time-varying position, possibly that of a particle, in the unprimed system. The coordinates of the same time-varying position in the primed system are

$$X'(t) = K[X(t) - vt]$$
$$Y'(t) = Y(t)$$
$$Z'(t) = Z(t)$$

To express these coordinates in terms of the time in the primed system, t', it is necessary to substitute

$$t = K[t' + (vX'/c^2)]$$

which, itself, involves the position, X'.

While solving for $X'(t')$, $Y'(t')$, and $Z'(t')$ given specific $X(t)$, $Y(t)$, and $Z(t)$ can be quite involved, comparison of the velocities in the two systems may be done easily and in general. Let

$$u_x = dX/dt, \quad u_y = dY/dt, \quad u_z = dZ/dt$$

and let

$$u'_x = dX'/dt', \quad u'_y = dY'/dt', \quad u'_z = dZ'/dt'$$

then

$$\begin{aligned} u'_x &= (u_x - v)/[1 - (vu_x/c^2)] \\ u'_y &= u_y/K[1 - (vu_x/c^2)] \\ u'_z &= u_z/K[1 - (vu_x/c^2)] \end{aligned} \tag{2.2}$$

The transformation from the primed to the unprimed quantities has the same form, with v replaced by $-v$.

The velocities u and u$'$ may be time varying; it is not required that they be constant.

Transformations of Net Charge, Mass, and Momentum. If charge is conserved in an inertial system, special relativity predicts that the net charge of a body does not depend upon its velocity (Einstein, 1905, section 9):

$$Q' = Q \tag{2.3}$$

If momentum is conserved in all inertial systems, special relativity predicts that the mass of a body depends upon its velocity according to

$$M = KM_0$$

where M_0 is the mass of the body when at rest (Møller, 1972, pp. 65–68).

To obtain the general transformation for net mass, consider a mass with velocity u in the unprimed system. In that system the mass of the body is

$$M = M_0/[1 - (u/c)^2]^{1/2}$$

while in the primed system, the mass of the same body is

$$M' = M_0/[1 - u'/c)^2]^{1/2}$$

giving

$$M' = K[1 - (vu_x/c^2)]M \tag{2.4}$$

Identifying the x momentum of the mass as $P_x = Mu_x$, the mass transformation (2.4) may be written as

$$M' = K[M - (v/c^2)P_x]$$

Similarly, the combination of (2.4) and (2.2) gives

$$P'_x = M'u'_x = K(u_x - v)M = K(P_x - vM)$$
$$P'_y = M'u'_y = Mu_y = P_y$$
$$P'_z = P_z$$

The linearity of these transformations, the transformation coefficients depending only upon the relative velocity of the inertial systems, v, makes them valid for a body consisting of a number of component masses (including, of course, any mass energy of interaction), each moving at different velocities.

3. *Transformations when the Body Has Uniform Velocity*

In this section, the transformations of charge density and mass density are developed for bodies traveling at uniform velocity in an inertial frame. This development is done easily and simply by employing Dirac delta functions to represent point charges and masses.

Since the results for charge and charge density are generally much more familiar than those involving mass density, the results of electromagnetics serve to demonstrate and give confidence in the method prior to its application to the more complicated case of distributed mass.

The need for a third quantity, stress density, in addition to mass density and momentum density is shown to arise quite naturally from the transformations themselves, rather than from *a priori* considerations involving forces or pressures. This leads, eventually, to a unique description for stress density.

Transformation of Charge and Current Densities. To determine how the charge density of a body must transform from one inertial system to another, consider the density of a point charge moving at *uniform* velocity u in the unprimed system. This charge density will be represented by Dirac delta functions (Zemanian, 1965). If, in the unprimed system, the point charge has net charge Q and is, for simplicity of notation, located at the origin at $t = 0$, the charge density is

$$\rho(x, y, z, t) = Q\delta(x - u_x t)\delta(y - u_y t)\delta(z - u_z t)$$

The same point charge will have density

$$\rho'(x', y', z', t') = Q'\delta(x' - u'_x t')\delta(y' - u'_y t')\delta(z' - u'_z y')$$

in the primed system. Substituting (2.1) for x', y', z', and t', (2.2) for u'_x, u'_y, and u'_z, and (2.3) for Q' gives

$$\rho' = Q\delta\{(x - u_x t)/K[1 - (vu_x/c^2)]\}\,\delta(y - u_y t)\delta(z - u_z t)$$

Since $\delta(ax) = \delta(x)/a$ if a is a positive constant, we have

$$\rho' = K[1 - (vu_x/c^2)]\,\rho$$

Identifying the current density components

$$J_x = \rho u_x, \quad J_y = \rho u_y, \quad J_z = \rho u_z$$

there results

$$\begin{aligned} \rho' &= K[\rho - (v/c^2)J_x] \\ J'_x &= \rho' u'_x = K(u_x - v)\rho = K(J_x - v\rho) \\ J'_y &= J_y \\ J'_z &= J_z \end{aligned} \tag{3.1}$$

The equations (3.1) relate the charge and current densities of a body moving with uniform velocity in one inertial system to the charge and current densities of the same body as viewed in another inertial system. Since they do not involve the velocity of the body directly, they are valid for any number of bodies, each of which is moving with various uniform velocities.

In section 4, it will be shown that a general distribution of charge and current may be decomposed into a set of component charge distributions, each of which has uniform velocity. Thus (3.1) becomes a general transformation, applicable to arbitrary charge and current distributions.

Transformations of Mass and Momentum Densities. To determine how the mass density of a body must transform from one inertial system to another, the same analysis will be performed as was done with charge density. The resulting transformations are not linear in mass and momentum density, a result which will give fundamental importance to the stress density tensor.

If, in the unprimed system, a point mass moving at uniform velocity with net mass M and uniform velocity u is at the origin at $t = 0$, the mass density is

$$m(x, y, z, t) = M\delta(x - u_x t)\delta(y - u_y t)\delta(z - u_z t)$$

The same point mass will have density

$$m'(x', y', z', t') = M'\delta(x' - u'_x t')\delta(y' - u'_y t')\delta(z' - u'_z t')$$

in the primed system. Substituting (2.1), (2.2), and (2.4) gives

$$\begin{aligned} m' &= MK[1 - (vux/c^2)]\,\delta\{(x - u_x t)/K[1 - (vu_x/c^2)]\} \\ &\quad \times \delta(y - u_y t)\delta(z - u_z t) \\ &= K^2[1 - (vu_x/c^2]^2 m \end{aligned} \tag{3.2}$$

Similarly, the transformation of the momentum density of a mass distribution is found to be

$$\begin{aligned} \rho'_x &= K^2[1 - (vu_x/c^2)]\,[1 - (v/u_x)]\,\rho_x \\ \rho'_y &= K[1 - (vu_x/c^2)]\,\rho_y \\ \rho'_z &= K[1 - (vu_x/c^2)]\,\rho_z \end{aligned} \tag{3.3}$$

Introduction of the Stress Density Tensor. The transformations of mass (3.6) and momentum (3.3) densities cannot be expressed as a linear combination of the quantities (with coefficients which are only functions of the relative coordinate velocity, v), although the mixed linear relation

$$m' + (v/c^2)\rho x' = m - (v/c^2)\rho_x$$

may be easily derived.

By defining

$$\begin{aligned} b_{11} &= m u_x{}^2 = \rho_x u_x \\ b_{21} &= m u_y u_x = \rho_y u_x \\ b_{31} &= m u_z u_x = \rho_z u_x \end{aligned}$$

however, the mass and momentum density transformations may be written as the linear equations

$$\begin{aligned} m' &= K^2[m - (2v/c^2)\rho_x + (v^2/c^4)b_{11}] \\ \rho'_x &= K^2\{[1 + (v/c)^2]\,\rho_x - (v/c^2)b_{11} - vm\} \\ \rho'_y &= K[\rho_y - (v/c^2)b_{21}] \\ \rho'_z &= K[\rho_z - (v/c^2)b_{31}] \end{aligned} \tag{3.4}$$

The b's also transform linearly:

$$\begin{aligned} b'_{11} &= K^2(b_{11} - 2v\rho_x + v^2 m) \\ b'_{21} &= K(b_{21} - v\rho_y) \\ b'_{31} &= K(b_{31} - v\rho_z) \end{aligned} \tag{3.5}$$

The other components of the dyad

$$2_b = \rho u = \begin{bmatrix} \rho_x u_x & \rho_x u_y & \rho_x u_z \\ \rho_y u_x & \rho_y u_y & \rho_y u_z \\ \rho_z u_x & \rho_z u_y & \rho_z u_z \end{bmatrix}$$

which would enter into a more general Lorentz transformation, transform linearly as

$$\begin{aligned} b'_{22} &= m' u_y'^2 = m u_y^2 = b_{22} \\ b'_{23} &= b'_{32} = m' u'_y u'_z = m u_y u_z = b_{23} = b_{32} \\ b'_{33} &= m' u_z'^2 = m u_z^2 = b_{33} \end{aligned} \tag{3.6}$$

Validity of these Results. Equations (3.4)-(3.6), analogous to (3.1), relate the mass, momentum, and stress densities of a body moving at uniform velocity in one inertial system to the same quantities as viewed in another inertial system. Since these equations do not involve the velocity of the body directly, they are valid for any number of bodies, each of which is moving at various uniform velocities.

In section 4 it will be shown that a general mass, momentum, and stress distribution may be decomposed into a set of component functions, each of which has uniform velocity. Thus (3.4)-(3.6) will become general transformations, applicable to arbitrary mass, momentum, and stress distributions.

4. *General Density Transformations*

In this section, the previously derived transformations, restricted to bodies moving with uniform velocities, are shown to be valid in general by demonstrating that the general functions may be decomposed into a set of uniform velocity component functions.

The method of decomposition of a general charge and current distribution into a set of uniform velocity distributions is due to Elliott (1966; pp. 150–152). It is here extended to distributions of mass, momentum, and stress, with interesting results, recognizing the necessity of linear transformation equations together with the proper continuity relations.

Decomposition of a General Charge and Current Distribution. The general validity of the transformations (3.1) is assured if an arbitrary set of charge and current densities may be decomposed into a set of charge distributions moving at various uniform velocities. Each member distribution of the set would be static in some Lorentz frame.

Let $*J(k_x, k_y, k_z, \omega)$ be the fourfold Fourier transform of a general current density function:

$$*J(k_x, k_y, k_z, \omega) = \iint \int_{-\infty}^{\infty} \int J(x, y, z, t)$$
$$\times\, e^{-i\,(k_x x + k_y y + k_z z + \omega t)}\, dx\, dy\, dz$$

Similarly, let $*\rho(k_x, k_y, k_z, \omega)$ be the transform of a general charge density function.

The continuity equation for charge and current densities (Stratton, 1941, p. 5, pp. 69–72),

$$\nabla \cdot J + \partial\rho/\partial t = 0 \tag{4.1}$$

requires that

$$*\rho = -\,(k \cdot *J)/\omega$$

where

$$k = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$$

and $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors in the x, y, and z directions, respectively.

If the charge and current densities in the interval $(dk, d\omega)$ are treated as an isolated component of J and ρ, then this distribution satisfies

$$dJ = (d\rho)u$$

where

$$u(k, \omega) = *J/*\rho = -\,(\omega *J)/(k \cdot *J).$$

This velocity is independent of x, y, z, and t and so is a common velocity shared by all parts of this fictitious component of J, ρ.

Some of these component charge and current densities have velocities in excess of the speed of light. While such components might not be regarded as being physical entities, they are quite admissible mathematically (Elliott, 1966, p. 152).

Thus (3.1) is a general transformation, not limited to charge distributions moving at uniform velocity.

Continuity Equations for Mass and Momentum. To the extent that an arbitrary mass distribution may be expressed as the sum of static mass distributions in various Lorentz frames, (3.4)–(3.6) may be used to transform a general mass and momentum distribution. The validity of such a general decomposition of mass and momentum densities is assured if proper continuity relations between m, p, and ^{2}b exist, as was the case with the charge and current densities, (4.1).

The continuity equation

$$\nabla \cdot p = -\partial m/\partial t \tag{4.2}$$

expresses the conservation of mass and is firmly established in present theory (Jammer, 1964).

The second continuity equation that we require is

$$\nabla \cdot {}^2\mathrm{b} = -\partial \mathrm{p}/\partial t \tag{4.3}$$

which expresses the conservation of momentum. It is equivalent to the three scalar equations

$$\nabla \cdot \mathrm{b}_x = -\partial \rho_x/\partial t$$

$$\nabla \cdot \mathrm{b}_y = -\partial \rho_y/\partial t$$

$$\nabla \cdot \mathrm{b}_z = -\partial \rho_z/\partial t$$

where

$$\mathrm{b}_x = b_{11}\hat{x} + b_{12}\hat{y} + b_{13}\hat{z}$$

$$\mathrm{b}_y = b_{21}\hat{x} + b_{22}\hat{y} + b_{23}\hat{z}$$

$$\mathrm{b}_z = b_{31}\hat{x} + b_{32}\hat{y} + b_{33}\hat{z}$$

This continuity equation, too, is well established in continuum theory (Lindsay, 1969), where the quantity ${}^2\mathrm{b}$ is known as the stress density tensor. Only the divergence of ${}^2\mathrm{b}$ is fixed, however, leaving as yet unanswered questions regarding its physical, measurable, existence. This situation is analogous to that with the Poynting vector of electromagnetic fields (Stratton, 1941, pp. 131-137), where any number of vector fields besides $\mathrm{E} \times \mathrm{H}$ might represent the flow of electromagnetic energy.

In a later section it will be demonstrated that the adoption of (3.4)-(3.6) as general transformations, not limited to masses traveling at uniform velocity, demands a specific ${}^2\mathrm{b}$, not just one with fixed divergence.

Transformation of the Continuity Equations. Because the continuity of momentum (4.3) is less familiar than the continuity of mass (4.2) to most and because both are of fundamental importance to the development to follow, their interrelationship is now examined. In particular, the transformation of these equations from one inertial system to another is used to demonstrate that continuity of mass implies continuity of momentum and vice versa.

Transformation of

$$\nabla' \cdot \mathrm{p}' + \partial m'/\partial t'$$

by (3.4)-(3.6) and (2.1) gives

$$K(\nabla \cdot \mathrm{p} + \partial m/\partial t) - (Kv/c^2)(\nabla \cdot \mathrm{b}_x + \partial \rho_x/\partial t) \tag{4.4}$$

Similarly,

$$\nabla' \cdot \mathrm{b}'_x + \partial \rho'_x/\partial t'$$

transforms as

$$K(\nabla \cdot \mathrm{b}_x + \partial \rho_x/\partial t) - K\nu(\nabla \cdot \mathrm{p} + \partial m/\partial t) \quad (4.5)$$

The combination of quantities

$$\nabla' \cdot \mathrm{b}'_y + \partial \rho'_y/\partial t'$$

transforms as

$$\nabla \cdot \mathrm{b}_y + \partial \rho y/\partial t$$

and

$$\nabla' \cdot \mathrm{b}_z + \partial \rho'_z/\partial t'$$

transforms as

$$\nabla \cdot \mathrm{b}_z + \partial \rho_z/\partial t$$

The continuity equations (4.2) and (4.3) are thus linked to one another under special relativity. If the continuity equation for mass is true in all inertial systems, the continuity equation for momentum necessarily also must hold in all inertial systems.

Decomposition of a General Mass Distribution. The general validity of the transformations of mass, momentum, and stress densities (3.4)-(3.6) is assured if an arbitrary set of these quantities may be decomposed into a set of component distributions moving at various constant velocities.

Let $*m$, $*\mathrm{p} = *\rho_x\hat{x} + *\rho_y\hat{y} + *\rho_z\hat{z}$, $*\mathrm{b}_x$, $*\mathrm{b}_y$, and $*\mathrm{b}_z$ be the fourfold Fourier transforms of the densities m, p, b_x, b_y, and b_z, respectively, associated with a general distribution of these quantities. For example,

$$*m(k_x, k_y, k_z, \omega) = \iint \int_{-\infty}^{\infty} \int m(x, y, z, t) \times e^{-i(k_x x + k_y y + k_z z + \omega t)}\, dx\, dy\, dz$$

The continuity equation for mass (4.2) requires that

$$*m = -(\mathrm{k} \cdot *\mathrm{p})/\omega \quad (4.6)$$

and continuity equation of momentum (4.3) requires that

$$\begin{aligned} *\rho_x &= -(\mathrm{k} \cdot *\mathrm{b}_x)/\omega \\ *\rho_y &= -(\mathrm{k} \cdot *\mathrm{b}_y)/\omega \\ *\rho_z &= -(\mathrm{k} \cdot *\mathrm{b}_z)/\omega \end{aligned} \quad (4.7)$$

If the various densities in an interval $(d\mathrm{k}, d\omega)$ are treated as an independent entity, for example

$$dm = (1/2\pi)\, *m e^{i(k_x x + k_y y + k_z z + \omega t)} \times dk_x\, dk_y\, dk_z\, d\omega \quad (4.8)$$

then those components of the overall densities satisfy

$$d\mathsf{p} = (dm)\mathsf{u}_1$$

where

$$\mathsf{u}_1(\mathsf{k}, \omega) = {}^*\mathsf{p}/{}^*m = -(\omega{}^*\mathsf{p})/(\mathsf{k} \cdot {}^*\mathsf{p})$$

and

$$d\mathsf{b}_x = (d\rho_x)\mathsf{u}_2$$

where

$$\mathsf{u}_2(\mathsf{k}, \omega) = {}^*\mathsf{b}_x/{}^*\rho_x = -(\omega{}^*\mathsf{b}_x)/(\mathsf{k} \cdot {}^*\mathsf{b}_x)$$

$$d\mathsf{b}_y = (d\rho_y)\mathsf{u}_3$$

where

$$\mathsf{u}_3(\mathsf{k}, \omega) = {}^*\mathsf{b}_y/{}^*\rho_y = -(\omega{}^*\mathsf{b}_y)/(\mathsf{k} \cdot {}^*\mathsf{b}_y)$$

$$d\mathsf{b}_z = (d\rho_z)\mathsf{u}_4$$

where

$$\mathsf{u}_4(\mathsf{k}, \omega) = {}^*\mathsf{b}_z/{}^*\rho_z = -(\omega{}^*\mathsf{b}_z)/(\mathsf{k} \cdot {}^*\mathsf{b}_z)$$

It remains to be shown that the four velocities defined above, u_1, u_2, u_3, and u_4, are identical or may be chosen to be identical and equal to the velocity of each of the moving distributions of fixed shape (4.8). Then, a general density of m, p, ${}^2\mathsf{b}$ may be decomposed into a set of uniform velocity mass distributions, all of which are related by equations (3.4)–(3.6).

A Specific Stress Density Tensor. The vector fields ${}^*\mathsf{b}_x$, ${}^*\mathsf{b}_y$, and ${}^*\mathsf{b}_z$ may each have any directions at each position in space and still maintain the relationships to *p required by (4.7). Choosing

$$\text{dir } {}^*\mathsf{b}_x = \text{dir } {}^*\mathsf{b}_y = \text{dir } {}^*\mathsf{b}_z = \text{dir } {}^*\mathsf{p} \tag{4.9}$$

gives

$$\mathsf{u}_1 = \mathsf{u}_2 = \mathsf{u}_3 = \mathsf{u}_4$$

The mass, momentum, and stress density transformations (3.4)–(3.6) become general relations when, of all possible ${}^2\mathsf{b}$ distributions satisfying the continuity relation (4.3), the field for which (4.9) is also satisfied is chosen.

Combining (4.7) and (4.9), there results

$${}^{*2}\mathsf{b} = ({}^*\rho{}^*\rho)/{}^*m \tag{4.10}$$

in which the Fourier transformed stress density tensor is expressed in terms of the momentum and mass density transforms.

If (4.10) does not hold, the transformations (3.4)-(3.6) must contain additional terms and/or there must be additional relations among m, p, and ^{2}b, as may be demonstrated by considering an accelerating mass in a manner similar to that given in section 3.

5. *Objective Function Formulation*

It is now shown that the Lorentz transformation of Fourier transformed quantities is of the same form as the space and time domain transformations.

New functions are then introduced so that the Lorentz transformations of mass, momentum, and stress density are simplified to a form identical to the transformations of electric charge and current density.

The new functions, one a scalar and one a vector, are shown to satisfy a continuity equation, a circumstance which gives rise to speculation as to the physical significance of these quantities in the final section.

Lorentz Transformations of Fourier Transforms. Consider a function that Lorentz-transforms as

$$f'(x', y', z', t') = Cf\{K[x' - (v/c^2)t'], y', z', K(t' - vx')\}$$

where C is a constant. The fourfold Fourier transform of the function f Lorentz-transforms as

$$*f'(x', y', z', t') = C*f\{K[k'_x - (v/c^2)\omega'], k'_y, k'_z, K(\omega' - vk'_x)\}$$

The Fourier-transformed quantity Lorentz-transforms in the same manner as the space and time domain in quantity, with

$$\begin{aligned} k'_x &= K(k_x - v\omega) \\ k'_y &= k_y \\ k'_z &= k_z \\ \omega' &= K[\omega - (v/c^2)k_x] \end{aligned} \tag{5.1}$$

Applying the above principle to (3.4)-(3.6) gives

$$\begin{aligned} *m' &= K^2[*m - (2v/c^2)*\rho_x + (v^2/c^4)*b_{11}] \\ *\rho'_x &= K^2\{[1 + (v/c)^2]*\rho_x - (v/c^2)*b_{11} - v*m\} \\ *\rho'_y &= K[*\rho_y - (v/c^2)*b_{21}] \\ *\rho'_z &= K[*\rho_z - (v/c^2)*b_{31}] \\ *b'_{11} &= K^2(*b_{11} - 2v*\rho_x + v^2*m) \\ *b'_{21} &= K(*b_{21} - v*\rho_y) \\ *b'_{31} &= K(*b_{31} - v*\rho_x) \\ *b'_{22} &= *b_{22} \end{aligned}$$

$$*b'_{23} = *b_{23}$$
$$*b'_{33} = *b_{33} \tag{5.2}$$

where the substitutions of variables (5.1) are understood.

Density Transformations in Terms of the Objective Functions. Examination of equations (5.2), using the relation (4.10), indicates the expediency of function changes for the purpose of simplification. Defining new functions, here called *objective functions* for the distribution of mass and momentum,

$$\begin{aligned} *e &= \sqrt{*m} \\ *\mathrm{a} &= *\mathrm{p}/\sqrt{*m} \end{aligned} \tag{5.3}$$

the Fourier transforms of the densities become

$$\begin{aligned} *m &= (*e)^2 \\ *\mathrm{p} &= *e*\mathrm{a} \end{aligned} \tag{5.4}$$

and, using (4.10),

$$*2_{\mathrm{b}} = *\mathrm{a}*\mathrm{a} \tag{5.5}$$

The algebraic sign in the definition of $*e$ is as yet ambiguous, but it will be advantageous to make symmetric selections so that e and a are real functions.

In terms of the objective functions, the density transforms (5.2) Lorentz-transform as

$$\begin{aligned}
*m' &= (*e')^2 = \{K[*e - (v/c^2)*a_x]\}^2 \\
*\rho'_x &= *a'_x*e' = \{K[*a_x - v*e]\}\{K[*e - (v/c^2)*a_x]\} \\
*\rho'_y &= *a'_y*e' = \{*a_y\}\{K[*e - (v/c^2)*a_x]\} \\
*\rho'_z &= *a'_z*e' = \{*a_z\}\{K[*e - (v/c^2)*a_x]\} \\
*b'_{11} &= *a'_x*a'_x = \{K[*a_x - v*e]\}\{K[*a_x - v*e]\} \\
*b'_{21} &= *a'_y*a'_x = \{*a_y\}\{K[*a_x - v*e] \\
*b'_{31} &= *a'_z*a'_x = \{*a_z\}\{K[*a_x - v*e] \\
*b'_{22} &= *a'_y*a'_y = \{*a_y\}\{*a_y\} \\
*b'_{23} &= *a'_y*a'_z = \{*a_y\}\{*a_z\} \\
*b'_{33} &= *a'_z*a'_z = \{*a_z\}\{*a_z\}
\end{aligned}$$

showing that these equations are expressed much more easily using the objective functions (5.3)-(5.5) and their transformations,

$$\begin{aligned}
*e' &= K[*e - (v/c^2)*a_x] \\
*a'_x &= K[*a_x - v*e] \\
*a'_y &= *a_y \\
*a'_z &= *a_z
\end{aligned} \tag{5.6}$$

The objective functions transform in the same manner as do the charge and current densities.

Objective Function Continuity. The continuity equation (4.6),

$$*m = -(\mathrm{k} \cdot *\mathrm{p})/\omega \tag{4.6}$$

is, in terms of the objective functions,

$$*e = -(\mathrm{k} \cdot *\mathrm{a})/\omega$$

giving

$$\nabla \cdot \mathrm{a} = -\partial e/\partial t$$

in the space and the time domain.

Thus the existence of a conserved fluid, related to the mass fluid, is established.

6. *Conclusion*

The Lorentz transformation of the mass and momentum densities of bodies moving at uniform velocity were derived by considering the density of a point mass.

To eliminate the restriction of uniform velocity and thus to develop a general transformation, a decomposition of a general mass distribution into uniform velocity components was attempted, paralleling the electromagnetic result. Such a decomposition was shown to be possible with the specific stress density tensor given by (4.10).

The choice (4.10) is not only sufficient; it is necessary unless the density transformations are yet more complicated than those given or there are additional relations among the quantities involved.

When the known stress density is substituted into the transformation equations, it becomes apparent that these equations may be greatly simplified by the introduction of the objective functions.

Further examinations of the objective functions leads to the result that they represent a conserved fluid, which gives rise to the following speculation: Are the objective functions in some way related to the electromagnetic properties of a body?

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